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TWO PROBLEMS ON ORDERABLE SEMIGROUPS

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I. A semigroup S is said to be an orderable semigroup or an o-semigroup if S admits a simple order to make it a simply ordered semigroup.

PROBLEM 1. Characterise right cancellative, right simple o-semigroups without idempotents. (This problem was proposed in our lecture note [6].)

In connection with the above problem we have the following two results.

RESULT 1. Let p, q be two infinite cardinals such that $q \leqq p$ and let $S(p,q)$ be a Baer-Levi semigroup of type (p,q). Then $S(p,q)$ is a right cancellative, right simple semigroup without idempotents but is not an o-semigroup.

The first assertion is given in [1] Theorem 8.2. Now by way of contradiction, we assume that $S(p,q)$ is an o-semigroup. Thus $S(p,q)$ can be considered as a simply ordered semigroup.

First suppose $p = q$. By definition, there exists a set A such that $|A| = p$ and $S(p.p)$ is the family of all injective mappings α of A into A with $|A\setminus\alpha A| = p$. Let B_1, B_2, B_3 be mutually disjoint subsets of A such that $|B_1| = |B_2| = |B_3| = p$ and $B_1 \cup B_2 \cup B_3 = A$. Then for $i = 1, 2, 3$, there exists an injective mapping α_i of A onto B_i. Without loss of generality, we assume $\alpha_1 \leqq \alpha_2 \leqq \alpha_3$ in the simply ordered semigroup $S(p,p)$.

Since $S(p,p)$ is simply ordered without idempotents, we have either $\alpha_2 < \alpha_2^2$ or $\alpha_2^2 < \alpha_2$. Suppose $\alpha_2 < \alpha_2^2$. Then, since $S(p,p)$ has no idempotents, it follows from [5] Lemma 2 that we have $\alpha_3 < \alpha_3^2$. We have $A\alpha_2^2 = B_2\alpha_2 \subseteq A\alpha_2 = B_2$ and $A\alpha_3^2 = B_3\alpha_3 \subseteq A\alpha_3 = B_3$. Moreover

$$p = |B_1| \leqq |B_1 \cup (B_2 \cup A\alpha_2^2)| \leqq |A| = p,$$
$$p = |B_1| \leqq |B_1 \cup (B_3 \cup A\alpha_3^2)| \leqq |A| = p,$$

and so $|B_1 \cup (B_2 \cup A\alpha_2^2)| = |B_1 \cup (B_3 \cup A\alpha_3^2)| = p$. Since p is an infinite cardinal, we can choose a mutually dosjoint sets C and D such that $C \cup D = B_1 \cup (B_2 \cup A\alpha_2^2)$ and $|C| = |D| = p$. Since $|B_1 \cup (B_3 \cup A\alpha_3^2)| = p = |C|$, there exists an injection γ of $B_1 \cup (B_3 \cup A\alpha_3^2)$ onto C. Now we define a mapping β by:

$$x\beta = \begin{cases} x\alpha_3^{-1} & \text{if } x \in A\alpha_3^2, \\ x\alpha_2 & \text{if } x \in B_2, \\ x\gamma & \text{if } x \in B_1 \cup (B_3 \cup A\alpha_3^2). \end{cases}$$

Then β is a injection of A into A and

$$|A \setminus A\beta| = |A \setminus (A\alpha_3 \cup B_2\alpha_2 \cup C)| = |D| = p$$

and so $\beta \in S(p,p)$. Moreover, for every $x \in A$, we have $x\alpha_2 \in A\alpha_2 = B_2$ and $x\alpha_3^2 \in A\alpha_3^2$ and so $x\alpha_2\beta = x\alpha_2^2$ and $x\alpha_3^2\beta = x\alpha_3^2\alpha_3^{-1} = x\alpha_3$. Hence $\alpha_2\beta = \alpha_2^2$ and $\alpha_3^2\beta = \alpha_3$. Since $\alpha_2 < \alpha_2^2$, we have $\alpha_2^2 \leqq \alpha_2^3$ and, since $S(p,p)$ has no idempotents, we have $\alpha_2^2 < \alpha_2^3$. Hence

$$\alpha_2^2 < \alpha_2^3 = \alpha_2\alpha_2^2 = \alpha_2(\alpha_2\beta) = \alpha_2^2\beta = (\alpha_2\beta)\beta = \alpha_2\beta^2$$

and so $\alpha_2 < \beta^2$. Hence by [5] Lemma 2, we have $\beta^2 < (\beta^2)^2 = \beta^4$. But $\beta^2 \leqq \beta$ would imply that $\beta^4 \leqq \beta^3 \leqq \beta^2$. Since $S(p,p)$ is simply ordered, we have $\beta < \beta^2$. Hence

$$\alpha_3\beta \leqq \alpha_3^2\beta \leqq \alpha_3^2\beta^2 = (\alpha_3^2\beta)\beta = \alpha_3\beta$$

and so $\alpha_3\beta = \alpha_3^2\beta$. Hence

$$\alpha_3 = \alpha_3^2\beta = \alpha_3(\alpha_3\beta) = \alpha_3(\alpha_3^2\beta) = \alpha_3^3\beta = \alpha_3(\alpha_3^2\beta) = \alpha_3^2,$$

which contradicts the assumption that $S(p,p)$ has no idempotents.

In the case where $\alpha_2^2 < \alpha_2$, we can deduce a contradiction in a similar way.

Next we consider a general $S(p,q)$. We take an arbitrary $\alpha \varepsilon S(p,q)$ and put $T = \{ \xi \varepsilon S(p,q);\ \alpha\xi = \alpha \}$. Since $S(p,q)$ is right simple, we have $\alpha S = S$ and so T is nonempty. If $\xi, \eta \varepsilon T$, then $\alpha(\xi\eta) = (\alpha\xi)\eta = \alpha\eta = \alpha$, $\xi\eta \varepsilon T$ and so T is a subsemigroup of $S(p,q)$. Since $\alpha \varepsilon S(p,q)$, α is an injection of a set A into A such that $|A| = p$ and $|A \setminus A\alpha| = q$. Also for $\xi \varepsilon S(p,q)$, $\xi \varepsilon T$ if and only if ξ induces the identity mapping on $A\alpha$. For each $\xi \varepsilon T$, we denote by $\bar{\xi}$ the restriction of ξ to $A \setminus A\alpha$. Since ξ is an injection of A into A which induces the identity mapping on $A\alpha$, $\bar{\xi}$ is an injection of $A \setminus A\alpha$ into $A \setminus A\alpha$. Moreover, since $|A \setminus A\alpha| = q$ and

$$|(A \setminus A\alpha) \setminus (A \setminus A\alpha)\xi| = |A \setminus A\xi| = q,$$

$\bar{T} = \{ \bar{\xi};\ \xi \varepsilon T \}$ is a Baer-Levi semigroup $S(q,q)$. Further the mapping of T onto $\bar{T}$ which maps ξ into $\bar{\xi}$ is an isomorphism of T onto $\bar{T}$. Now since $S(p,q)$ is an o-semigroup, the subsemigroup T of $S(p,q)$ is also an o-semigroup. Hence $\bar{T} = S(q,q)$ is an o-semigroup, which contradicts the fact proved above.

RESULT 2. *There really exists a right cancellative, right simple o-semigroup without idempotents.*

In fact, let S be the set of all realvalued continuous functions α defined on the closed interval $[0,1]$, satisfying the conditions that $0 < 0\alpha$, $1\alpha < 1$ and the graph of α can be represented by a finite number of strictly increasing segments. It can be proved that S is a semigroup under the operation of composite of mappings and the semigroup S is right cancellative, right simple and has no idempotents (cf. [3]). Also it can be shown that S is a simply ordered semigroup under the order defined by:

for $\alpha, \beta \in S$, $\alpha < \beta$ if and only if there exist real numbers c and δ such that $0 \leqq c < 1$, $\delta > 0$, $x\alpha = x\beta$ for every $0 \leqq x < c$ but $x\alpha < x\beta$ for every $c < x < c + \delta$.

II. RESULT 3. The collection of all idempotent o-semigroups does not form a variety.

In fact, let L be a left zero semigroup and let R be a right zero semigroup. Then it can be checked that, with respect to an arbitrary simple order on L, L is a simply ordered semigroup and, with respect to an arbitrary simple order on R, R is a simply ordered semigroup. Hence L and R are o-semigroups. In particular, if $|L| \geqq 2$ and $|R| \geqq 2$ and if S is the direct product semigroup of L and R, then S is a rectangular band which is neither a left zero semigroup nor a right zero semigroup. Hence by [4] Theorem 1, S is not an o-semigroup. Hence the collection of all idempotent o-semigroups is not closed with respect to the formation of direct products and so is not a variety.

Since the intersection of a family of varieties of semigroups is a variety of semigroups, we can consider a variety of semigroups which is generated by idempotent o-semigroups.

In connection with this, we give the following problem.

PROBLEM 2. Give the concrete description of the variety of semigroups generated by idempotent o-semigroups.

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